we obtain a nonlinear model of a couple-free continuum in a formulation unrestricted by the condition of smallness of the local rotations and thereby generalizing the formulation of Biot [2].

Thus, the proposed technique for the construction of nonlinear models of deformable media provides a unified kinematic foundation for couple-stress and couple-free media in an ultimately simple (vector) representation.

## LITERATURE CITED


1. R. A. Toupin, "Theories of elasticity with couple-stresses," Arch. Rat. Mech., Anal., 17, 85-112 (1964).
2. M. A. Biot, "Nonlinear theory of elasticity and the linearized case for a body under initial stress," Philos. Mag., Ser. 7, 27, No. 183 (1939).
3. A. I. Lur'e, Analytical Mechanics [in Russian], Fizmatgiz, Moscow (1961).
4. W. Nowacki, Theory of Elasticity [Russian translation], Mir, Moscow (1975).
5. W. T. Koiter, "Couple stresses in the theory of elasticity," Nederl. Akad. Wetensch. Proc., Ser. B., 67, 17-29; 30-34 (1964).


# ASYMPTOTIC BEHAVIOR OF BOUNDARY-VALUE PROBLEMS FOR AN ELASTIC RING REINFORCED WITH VERY RIGID FIBERS


**Yu. A. Bogan**


UDC 539.3

The following boundary-value problems are investigated for an elastic ring reinforced with very rigid fibers arranged in concentric circles: a) The stresses are given at the boundary; b) the bending deflection and angle of rotation are given at the boundary. The generalized Hooke's law [1] is adopted as the initial governing equations; as a consequence, the final results are valid for standard models of a composition elastic model [2, 3].

We formulate asymptotic representations of the solutions of boundary-value problems a) and b) on the assumption that the rigidity of the material in the circumferential direction is much greater than the shear rigidity.

We show that a boundary layer sets in along the boundary; in case a) the boundary conditions for the limiting boundary-value problem do not coincide with any of the boundary conditions for the sublimiting problem. Problem b) degenerates into the limiting problem in a regular manner.

1. Let us consider problem a). We assume that the elastic ring is cylindrically orthotropic, and we apply the generalized Hooke's law in the form [$(r, \theta)$ denotes polar coordinates]

$$\sigma_r = c_{11}\varepsilon_r + c_{12}\varepsilon_\theta,\ \sigma_\theta = c_{12}\varepsilon_r + c_{22}\varepsilon_\theta,\ \tau_{r\theta} = c_{66}\gamma_{r\theta}.$$

We introduce the dimensionless stresses and rigidities, setting

$$\bar\sigma_r = \sigma_r c_{66}^{-1},\quad \bar\sigma_\theta = \sigma_\theta c_{66}^{-1},\quad \bar\tau_{r\theta} = \tau_{r\theta} c_{66}^{-1},\quad d_{ij} = c_{ij} c_{66}^{-1},\quad i, j = 1, 2,$$

and in all that follows we retain the same notation as before for the dimensionless stresses. Let $d_{22} \gg 1$; in real situations this relation holds for an elastic ring reinforced with one very rigid set of fibers r = const. We put $\varepsilon^2 = d_{22}^{-1}$, $d = d_{11}^{-1}$, $c = d_{12}^2 + 2d_{12}$, $t = \ln r$. Then the equation for the stress function $w(t, \theta)$ can be written in the form

$$\varepsilon^2 N(w) + M(w) = 0 \tag{1.1}$$

on the assumption that mass forces are absent. In (1.1)

$$N(w) = \frac{\partial^4 w}{\partial t^4} - 4\frac{\partial^3 w}{\partial t^3} + 5\frac{\partial^2 w}{\partial t^2} + 2\frac{\partial w}{\partial t} + 2dc\frac{\partial^3 w}{\partial t\partial\theta^2} - dc\frac{\partial^4 w}{\partial t^2\partial\theta^2} - dc\frac{\partial^2 w}{\partial\theta^2},$$

$$M(w) = d\frac{\partial^4 w}{\partial\theta^4} + \frac{\partial^4 w}{\partial t^2\partial\theta^2} - 2\frac{\partial^3 w}{\partial t\partial\theta^2} - d\frac{\partial^2 w}{\partial t^2} + 2d\frac{\partial w}{\partial t} + (1 + 2d)\frac{\partial^2 w}{\partial\theta^2}.$$


Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, No. 6, pp. 118-122, November-December, 1980. Original article submitted October 29, 1979.

For $r = a$, $r = b$, $0 < a < b$, we specify the boundary conditions

$$\sigma_r(a, \theta) = p_1(\theta),\ \sigma_r(b, \theta) = p_3(\theta),\ \tau_{r\theta}(a, \theta) = p_2(\theta),\quad \tau_{r\theta}(b, \theta) = p_4(\theta). \tag{1.2}$$

We recall that

$$\sigma_r = \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2},\quad \tau_{r\theta} = \frac{1}{r^2}\frac{\partial w}{\partial \theta} - \frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta}.$$

We put $t_0 = \ln a$, $t_1 = \ln b$, and

$$L_1(w) = \partial w/\partial t + \partial^2 w/\partial \theta^2,\ L_2(w) = \partial w/\partial \theta - \partial^2 w/\partial t \partial \theta.$$

The boundary conditions (1.2) can be written in the form

$$L_1(w)(t_0) = e^{2t_0}p_1(\theta),\quad L_2(w)(t_0) = e^{2t_0}p_2(\theta),$$
$$L_1(w)(t_1) = e^{2t_1}p_3(\theta),\quad L_2(w)(t_1) = e^{2t_1}p_4(\theta). \tag{1.3}$$

The boundary conditions in the form (1.3) are unsuitable for asymptotic analysis, because the derivative with respect to t enters into all the boundary conditions. We reduce the boundary conditions to a more useful form. We put

$$Pw = w + \frac{\partial^2 w}{\partial \theta^2},\quad g_k(\theta) = p_k(\theta) + \int_0^\theta p_{k+1}(u)\,du,\quad k = 1, 3,$$
$$g_k(\theta) = p_{k-1}(\theta) - p_k'(\theta),\quad k = 2, 4.$$

We then obtain

$$Pw(t_0) = e^{2t_0}g_1(\theta),\quad Pw(t_1) = e^{2t_1}g_3(\theta); \tag{1.4}$$
$$\frac{\partial}{\partial t}Pw(t_0) = e^{2t_0}g_2(\theta),\quad \frac{\partial}{\partial t}Pw(t_1) = e^{2t_1}g_4(\theta). \tag{1.5}$$

Thus, we derive the boundary conditions (1.4) at $t = t_0$. For this purpose we integrate the boundary conditions $L_2(w)(t_0) = e^{2t_0}p_2(\theta)$ with respect to $\theta$ and add the result to the first condition; we then obtain the first condition (1.4). Differentiating the boundary condition $L_2(w)(t_0) = e^{2t_0}p_2(\theta)$ with respect to $\theta$ and adding the result to the first condition, we obtain the first boundary condition (1.5). The procedure is analogous at $t = t_1$.

We now construct an asymptotic representation of the solution of the boundary-value problem (1.1), (1.2) for small $\epsilon$. For $\epsilon = 0$ Eq. (1.1) goes over to the equation $M(w) = 0$, which is of the composition type [4] with a double family of real characteristics r = const. Variation of the type of Eqs. (1.1) in the limit is such that a boundary-layer effect is observed along the boundaries $r = a$, b in connection with the rapid variation of the solution along the normal to the boundary in its immediate vicinity. The boundary-value problem (1.1), (1.2) has a unique solution in the event of self-equalization of the load [1].

We seek an asymptotic representation of the solution of the boundary-value problem (1.1), (1.4), (1.5) for small $\epsilon$ by means of two iteration processes [5].

**First Iteration Process.** We seek an approximate solution of the boundary-value problem (1.1), (1.4), (1.5) in the form

$$w_{N_1}(t, \theta) = \sum_{n=0}^{N_1} \varepsilon^n w_n(t, \theta). \tag{1.6}$$

Substituting (1.6) into (1.1) and (1.4), we obtain the recursive systems of equations

$$M(w_0) = 0,\quad Pw_0(t_0) = e^{2t_0}g_1(\theta),\quad Pw_0(t_1) = e^{2t_1}g_3(\theta); \tag{1.7}$$

$$M(w_1) = 0,\ M(w_{n+2}) + N(w_n) = 0,\ n \geqslant 0. \tag{1.8}$$

The boundary-value problem (1.7) represents the limiting boundary-value problem for the initial problem. The boundary conditions for the system (1.8) will be set forth later. The function $w_{N_1}(t, \theta)$, generally speaking, does not satisfy the boundary conditions (1.5), so that near $t = t_0$ and $t = t_1$ it is necessary to construct functions of the boundary-layer type, eliminating the discrepancy in the satisfaction of the boundary conditions (1.5).

**Second Iteration Process.** We construct boundary-layer functions near $t = t_0$ (they are similarly constructed near $t = t_1$). Accordingly, we extend the coordinate t near $t = t_0$, putting $\varepsilon\eta = t - t_0$. Then Eq. (2) acquires the form

$$\sum_{k=-2}^{2} \varepsilon^k P_k(w) = 0. \tag{1.9}$$

The explicit form of all the differential operators $P_k(w)$, $k = -2, -1, 0, 1, 2$ is immaterial for the ensuing discussion; we give only the operator $P_{-2}(w)$:

$$P_{-2}(w) = \frac{\partial^4 w}{\partial \eta^4} + \frac{\partial^4 w}{\partial \eta^2 \partial \theta^2} - d\frac{\partial^2 w}{\partial \eta^2}. \tag{1.10}$$

The differential operator $P_{-2}(w)$ in (1.10) is of the composition type. We seek an approximate solution of Eq. (1.9) in the form

$$\bar{w}_{N_1}(\eta, \theta) = \varepsilon \sum_{n=0}^{N_1 - 1} \varepsilon^n w_n^0(\eta, \theta). \tag{1.11}$$

Substituting (1.11) into (1.9), we obtain the recursive systems of equations

$$\begin{gathered} P_{-2}w_0^0 = 0, \quad P_{-2}w_1^0 + P_{-1}w_0^0 = 0, \quad P_{-2}w_2^0 + P_{-1}w_1^0 + P_0 w_0^0 = 0, \\ P_{-2}w_3^0 + P_{-1}w_2^0 + P_0 w_1^0 + P_1 w_0^0 = 0, \\ P_{-2}w_n^0 + P_{-1}w_{n-1}^0 + P_0 w_{n-2}^0 + P_1 w_{n-3}^0 + P_2 w_{n-4}^0 = 0, \quad n \geqslant 4. \end{gathered} \tag{1.12}$$

The boundary conditions for determination of the boundary-layer functions have the form

$$\begin{gathered} \left.\frac{\partial P w_n^0}{\partial \eta}\right|_{\eta=0} = -\left.\frac{\partial P w_n}{\partial t}\right|_{t=t_0}, \quad n \geqslant 1, \quad w_n^0(0, \eta) = w_n^0(2\pi, \eta), \quad n \geqslant 0, \\ \left.\frac{\partial P w_0^0}{\partial \eta}\right|_{\eta=0} = e^{2t_0} g_2(\theta) - \left.\frac{\partial P w_0}{\partial t}\right|_{t=t_0}, \quad w_n^0(\theta, +\infty) = 0, \quad n \geqslant 0. \end{gathered} \tag{1.13}$$

Constructing the functions $w_n^1(\eta_1, \theta)$ near $t = t_1$, we find that the initial boundary-value problem (1.1), (1.4), (1.5) has the asymptotic expansion

$$w(t, \theta) = \sum_{n=0}^{N_1} \varepsilon^n w_n(t, \theta) + \varepsilon \sum_{n=0}^{N_1 - 1} \varepsilon^n \left(w_n^0(\eta, \theta) + w_n^1(\eta_1, \theta)\right) + \varepsilon^{N_1+1} R_{N_1}(t, \theta), \tag{1.14}$$

in which $w_n^0(\eta, \theta)$ and $w_n^1(\eta_1, \theta)$ are boundary-layer functions near $t = t_0$ and $t = t_1$, respectively, $\varepsilon\eta_1 = t_1 - t$ and $\varepsilon^{N_1+1} R_{N_1}(t, \theta)$ is the remainder term. Substituting (1.14) into (1.4) and (1.5), we obtain the boundary conditions for $w_n(t, \theta)$, $n \geqslant 1$:

$$\begin{gathered} P w_n(t_0, \theta) = -P w_{n-1}^0(0, \theta) - P w_{n-1}^1(\eta_1(t_0), \theta), \\ P w_n(t_1, \theta) = -P w_{n-1}^0(\eta(t_1), \theta) - P w_{n-1}^1(0, \theta), \\ w_n(0, t) = w_n(2\pi, t). \end{gathered} \tag{1.15}$$

The boundary conditions (1.15) can be used to determine all the functions $w_n(t, \theta)$ in succession.

2. Certain remarks are in order regarding the asymptotic representations derived above. Unlike the initial boundary-value problem, the equations for the truncated boundary-value problem and boundary layer admit a solution in explicit form. Thus, let us suppose, for example, that the functions $g_k(\theta)$, $k = 1, 3$, are even functions of the polar angle $\theta$, and let us put

$$g_k(\theta) = \sum_{n=2}^{\infty} g_{kn} \cos n\theta, \quad k = 1,3,$$

whereupon

$$w_0(t, \theta) = \sum_{n=2}^{\infty} \frac{e^t}{1 - n^2}\left[f_{1n}(t) e^{t_0} + f_{3n}(t) e^{t_1}\right] \cos n\theta, \tag{2.1}$$

where

$$f_{1n}(t)=g_{1n}\frac{\operatorname{sh}\mu_n(t_1-t)}{\operatorname{sh}\mu_n(t_1-t_0)};\quad f_{3n}(t)=g_{3n}\frac{\operatorname{sh}\mu_n(t-t_0)}{\operatorname{sh}\mu_n(t_1-t_0)};$$
$$\mu_n=d^{1/2}(n^2-1)(n^2+d)^{-1/2}.$$

Let us determine, for example, the function $w_0^0(\eta, \theta)$. To do so we must solve the boundary-value problem

$$\frac{\partial^4 w_0^0}{\partial\eta^4}+\frac{\partial^4 w_0^0}{\partial\eta^2\partial\theta^2}-d\frac{\partial^2 w_0^0}{\partial\eta^2}=0,$$
$$\left.\frac{\partial P w_0^0}{\partial\eta}\right|_{\eta=0}=g(\theta),\quad w_0^0(\theta,+\infty)=0,\quad w_0^0(0,\eta)=w_0^0(2\pi,\eta). \tag{2.2}$$

An explicit solution of the boundary-value problem (2.2) is given by the equation

$$w_0^0(\eta,\theta)=-\sum_{n=2}^{\infty}\frac{e^{-\lambda_n\eta}}{\lambda_n(1-n^2)}(a_n\cos n\theta+b_n\sin n\theta),$$

where $\lambda_n=(n^2+d)^{1/2}$ and $a_n$ and $b_n$ are the Fourier coefficients of the function $g(\theta)$. Assuming that continuous derivatives of sufficiently high order exist for the boundary data, we can differentiate the expansion (1.14) and obtain asymptotic expansions for the stresses and displacements. Considering the axisymmetrical case ($\tau_{r\theta}=0$) separately, for the function $w(t, \theta)$ we obtain an ordinary differential equation, for which the boundary-value problem is solvable in explicit form; here the zeroth term in the expansion (1.14) is simply zero, and so the solution of the limiting boundary-value problem does not obey any of the initial boundary conditions (1.2).

A specific consideration is the fact that in the limit $\epsilon_\theta=0$ there also exists the limit $q_0=\lim_{\varepsilon\to+0}\varepsilon^{-2}\varepsilon_\theta$, a Lagrange multiplier, which is induced by the fact of inextensibility in the circumferential direction in the limit. The governing relations have the following form in the limit:

$$\sigma_r=d_{11}\varepsilon_r,\ \sigma_\theta=d_{12}\varepsilon_r+q_0,\ \ \tau_{r\theta}=\gamma_{r\theta},\ \ \varepsilon_\theta=0.$$

3. We now consider problem b) for Eq. (1.1). Let

$$w(t_0)=p_1(\theta),\quad w(t_1)=p_3(\theta),\quad \frac{\partial w}{\partial t}(t_0)=e^{t_0}p_2(\theta),\ \frac{\partial w}{\partial t}(t_1)=e^{t_1}p_4(\theta), \tag{3.1}$$

where the functions $p_k(\theta)$, k = 1, 2, 3, 4, are periodic functions of $\theta$. An asymptotic solution of the boundary-value problem (1.1), (3.1) is much simpler to construct than in the preceding case; it is given by expression (1.14), where it is required to put Pw = w in all the equations of 1, beginning with (1.4). The limiting boundary-value takes the form

$$Mw=0,\ w(t_0)=p_1(\theta),\ w(t_1)=p_3(\theta),$$

i.e., the degeneration of the initial boundary-value problem into the limiting problem is regular [5]. The absence of regularity of degeneration in the boundary-value problem a) is attributable to the fact that the boundary conditions (1.2) are "of the same order" with respect to $\epsilon$, since the boundary conditions (1.2) involve derivatives with respect to the normal coordinate, along which rapid variation of the solution takes place. To obtain a regularly degenerating boundary-value problem it is necessary to reduce the boundary conditions (1.2) to canonical form, as we have done above. The above-noted "confinement" of the boundary conditions (irregularity of degeneration) is typical of degenerate boundary-value problems [6, 7].

## LITERATURE CITED


1. S. G. Lekhnitskii, Theory of Elasticity of Anisotropic Bodies [in Russian], Nauka, Moscow (1977).
2. Yu. V. Nemirovskii, "Elastoplastic behavior of a reinforced layer," Zh. Prikl. Mekh. Tekh. Fiz., No. 6 (1969).
3. V. V. Bolotin, "Planar elasticity problem for parts made of reinforced materials," in: Strength Calculations [in Russian], No. 12, Mashinostroenie, Moscow (1966).
4. G. I. Éskin, "Boundary-value problems for equations with constant coefficients on the plane," Mat. Sb., 59(101) (1962).
5. M. I. Vishik and L. A. Lyusternik, "Regular degeneracy and boundary layer for linear differential equations with a small parameter," Usp. Mat. Nauk, 12, No. 5 (1957).
6. H. Brezis, "Singular perturbations of hyperbolic systems," in: Lecture Notes in Mathematics, No. 384, Springer, Berlin–New York (1974).
7. Yu. M. Vakhromeev and V. M. Kornev, "Boundary-value problems with a small parameter for ordinary differential equations," Differents. Uravn., 13, No. 7 (1977).